

A Short Note on Quenching Phenomena for Semilinear Parabolic Equations*

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In this paper, we investigate initial-boundary value problems for semilinear parabolic differential equations with singular term. A criterion for the appearance of quenching phenomena of classical solution to the above problems on a bounded domain is given and a global existence and nonexistence results of the above

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1. INTRODUCTION

Consider the initial-boundary value problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = g(u), & \text{in } \Omega \times (0, T), \\ u = 0, & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = \varphi(x), & \text{in } \Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset R^N$ is an N -dimensional domain, $\partial\Omega$ is the boundary of Ω , $\Delta = \sum_{i=1}^N \partial^2 / \partial x_i^2$ is the Laplace operator on Ω , $\varphi(x)$ is a nonnegative continuous function on Ω with $\sup_{x \in \Omega} \varphi(x) < b$, and $\varphi(x) \equiv 0$ on $\partial\Omega$, $g(s): [0, b) \mapsto (0, \infty)$ satisfies

(G1) $g(s)$ is locally Lipschitz on $[0, b)$ and $g(0) > 0$,

(G2) $\lim_{s \rightarrow b^-} g(s) = +\infty$.

Because of the singularity of the right-hand side of the equation, the classical solution of problem (1.1) is always connected with a so-called “quenching” phenomenon. The study of quenching phenomena began in 1975 with a paper [1] by Kwawarada, and since then, it has attracted much attention. For a detailed survey, readers can consult papers [2, 3] by Levine. Here we give only the main result of [4]. For convenience, we introduce the following definition first.

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DEFINITION. Let $u(x, t)$ be a classical solution of problem (1.1). We say that $u(x, t)$ quenches in finite time if there exists a real number $T \in (0, +\infty)$ such that $\lim_{t \rightarrow T^-} \sup_{x \in \Omega} u(x, t) = b$.

With this definition, the main result of [4], in our notation, can be stated as follows.

THEOREM A [4]. *Let $\Omega \subset R^N$ be a bounded domain, let $u(x, t)$ be the classical solution of problem (1.1), and let $g(s)$ satisfy (G1) and (G2). Then*

- (i) $u(x, t)$ exists globally for Ω small enough;
- (ii) $u(x, t)$ quenches in finite time for Ω large enough.

Based on the above result, our two basic questions are the following:

Question 1. In what sense can the largeness of the domain Ω ensure the appearance of quenching?

Question 2. If Ω is an unbounded domain, can the nonnegative classical solution of (1.1) exist globally?

On appearance, question 2 seems to be unreasonable because, by Theorem A, the classical solution of problem (1.1) must be quenching in finite time if Ω is only large enough, let alone Ω unbounded. But this is not the case. What surprises us is just that the nonnegative classical solution of problem (1.1) can still exist globally for some kinds of unbounded domains, though it must be quenching in finite time for most unbounded domains.

The paper is arranged as follows. Section 2 gives a criterion for the appearance of quenching phenomenon by using the first eigenvalue of the Laplace operator. In Sections 3 and 4, we study the existence and non-existence of a global solution of problem (1.1) on unbounded domains.

2. A CRITERION FOR THE APPEARANCE OF QUENCHING

Let $\Omega \subset R^N$ be a bounded domain. In this and the following sections, $\lambda_1(\Omega)$ and $\Psi_1(x)$ denote the first eigenvalue and the first eigenfunction of the following eigenvalue problem,

$$\begin{cases} \Delta \Psi + \lambda \Psi = 0, & \text{in } \Omega, \\ \Psi = 0, & \text{on } \partial\Omega. \end{cases}$$

For convenience, we choose $\Psi_1(x)$ so that

$$\Psi_1(x) > 0 \quad \text{in } \Omega, \quad \int_{\Omega} \Psi_1(x) dx = 1,$$

and sometimes denote $\lambda_1(\Omega)$ simply by λ_1 . Furthermore, if we assume that there are constants $c_1 > 0$ and c_2 such that

$$(G3) \quad g(s) \geq c_1 + c_2 s \text{ for } s \in [0, b],$$

$$(G4) \quad c_2 + (c_1/b) > 0,$$

then we have

THEOREM 2.1. *Let $\Omega \subset R^N$ be a bounded domain and let $u(x, t)$ be the classical solution of problem (1.1). If $g(s)$ satisfies (G1)–(G4), $\lambda_1 < c_2 + (c_1/b)$, then $u(x, t)$ must be quenching in a finite time T_{\max} , and for T_{\max} the estimate is*

$$\int_M^b \frac{ds}{g(s)} \leq T_{\max} \leq \frac{1}{c_2 - \lambda_1} \ln \frac{c_1 + (c_2 - \lambda_1)b}{c_1 + (c_2 - \lambda_1)m},$$

where $M = \sup_{x \in \Omega} \varphi(x) < b$, $m = \int_{\Omega} \varphi(x) \Psi_1(x) dx$, is valid.

Proof. Let $(0, T_{\max})$ be the maximum time interval in which the classical solution $u(x, t)$ of problem (1.1) exists. By (G1) and the comparison principle one has

$$0 < u(x, t) \quad \text{in } \Omega \times (0, T_{\max}).$$

Since b is a singular point of $g(s)$ and T_{\max} is maximal, one can conclude that if $T_{\max} < +\infty$, then

$$\lim_{t \rightarrow T_{\max}^-} \sup_{x \in \Omega} u(x, t) = b.$$

Otherwise, $u(x, t)$ can be extended beyond T_{\max} . This is impossible. Now, to prove Theorem 2.1, it is sufficient to prove that T_{\max} is finite and

$$\int_M^b \frac{ds}{g(s)} \leq T_{\max} \leq \frac{1}{c_2 - \lambda_1} \ln \frac{c_1 + (c_2 - \lambda_1)b}{c_1 + (c_2 - \lambda_1)m}.$$

To this end, multiplying the differential equation in (1.1) by $\Psi_1(x)$ and integration on Ω with respect to x , we have

$$\frac{d}{dt} \int_{\Omega} u \Psi_1 dx + \lambda_1 \int_{\Omega} u \Psi_1 dx = \int_{\Omega} g(u) \Psi_1 dx. \quad (2.1)$$

Since $u(x, t)$ is the classical solution of problem (1.1), one has

$$0 < u(x, t) < b \quad \text{for } x \in \Omega \quad t \in (0, T_{\max}).$$

Hence, by (G3) one has

$$g(u) \geq c_1 + c_2 u \quad \text{for } x \in \Omega \quad t \in (0, T_{\max}). \quad (2.2)$$

Substituting (2.2) into (2.1), one can obtain that

$$\frac{d}{dt} \int_{\Omega} u \Psi_1 dx + \lambda_1 \int_{\Omega} u \Psi_1 dx \geq c_1 + c_2 \int_{\Omega} u \Psi_1 dx. \quad (2.3)$$

Set $y(t) = \int_{\Omega} u \Psi_1 dx$, and (2.3) can be read as

$$\frac{dy}{dt} \geq c_1 + (c_2 - \lambda_1) y \quad \text{for } t \in (0, T_{\max}). \quad (2.4)$$

Since $0 < y(t) = \int_{\Omega} u \Psi_1 dx < b$ for $t \in (0, T_{\max})$, from the condition $\lambda_1 < c_2 + (c_1/b)$, i.e., $c_1 + (c_2 - \lambda_1)b > 0$, one has

$$c_1 + (c_2 - \lambda_1)y > 0 \quad \text{for } t \in (0, T_{\max}). \quad (2.5)$$

Taking into account (2.4) and (2.5), one has

$$\frac{dy}{c_1 + (c_2 - \lambda_1)y} \geq dt,$$

and this implies that

$$t \leq \frac{1}{c_2 - \lambda_1} \ln \frac{c_1 + (c_2 - \lambda_1)y(t)}{c_1 + (c_2 - \lambda_1)y(0)}. \quad (2.6)$$

Let $t \rightarrow T_{\max}$. From (2.6) it follows that

$$T_{\max} \leq \frac{1}{c_2 - \lambda_1} \ln \frac{c_1 + (c_2 - \lambda_1)y(T_{\max})}{c_1 + (c_2 - \lambda_1)y(0)}. \quad (2.7)$$

Due to

$$0 < y(T_{\max}) = \int_{\Omega} u(x, T_{\max}) \Psi_1 dx \leq b,$$

from (2.7), one can obtain that

$$T_{\max} \leq \frac{1}{c_2 - \lambda_1} \ln \frac{c_1 + (c_2 - \lambda_1)b}{c_1 + (c_2 - \lambda_1)m}, \quad (2.8)$$

where $m = y(0) = \int_{\Omega} \varphi(x) \Psi_1(x) dx$. It is obvious that

$$0 < \frac{1}{c_2 - \lambda_1} \ln \frac{c_1 + (c_2 - \lambda_1)b}{c_1 + (c_2 + \lambda_1)m} < +\infty.$$

Thus $u(x, t)$ quenches in finite time.

To obtain a lower bound of T_{\max} , let us consider the initial value problem of ODE

$$\begin{cases} \frac{d\eta(t)}{dt} = g(\eta), \\ \eta(0) = M, \end{cases} \quad (2.9)$$

where $M = \sup_{x \in \Omega} \varphi(x) < b$.

Since $g(s) > 0$, by (2.9) one has

$$\int_M^{\eta(t)} \frac{ds}{g(s)} = t. \quad (2.10)$$

Let t^* be the time for which $\lim_{t \rightarrow t^*} \eta(t) = b$, and from (2.10) we have

$$t^* = \int_M^b \frac{ds}{g(s)}. \quad (2.11)$$

Obviously, $\eta(t)$ is a superfunction of $u(x, t)$, and thus

$$T_{\max} \geq t^* = \int_M^b \frac{ds}{g(s)}. \quad (2.12)$$

This completes the proof of Theorem 2.1.

Remark. If $g(s) = 1/(1-s)^\beta$, and β is a positive constant, then conditions (G1)–(G4) are satisfied with $b=1$, $c_1=1$, and $c_2=\beta$. Moreover, if $g(s) \in C^2[0, b)$, $g''(s) \geq 0$, and $g'(0) + (g(0)/b) > 0$, then conditions (G3) and (G4) are satisfied and $c_1 = g(0)$, $c_2 = g'(0)$. In fact, by Taylor's expansion theorem, there is a number $\xi \in (0, s)$ such that

$$g(s) = g(0) + g'(0)s + \frac{1}{2}g''(\xi)s^2 \quad \text{for } s \in [0, b).$$

Since $g''(\xi) \geq 0$, we have

$$g(s) \geq g(0) + g'(0)s \quad \text{for } s \in [0, b).$$

This implies that (G3) is satisfied.

Theorem 2.1 and the above remark lead us to the following.

COROLLARY. *Assume that $g(s) \in C^2[0, b]$ satisfies (G2) and that $g'(0) + (g(0)/b) > 0$, $\lambda_1 < g'(0) + (g(0)/b)$. If $u(x, t)$ is the classical solution of problem (1.1), then $u(x, t)$ quenches in finite time T_{\max} , and for T_{\max} we have the estimate*

$$\int_M \frac{ds}{g(s)} \leq T_{\max} \leq \frac{1}{g'(0) - \lambda_1} \ln \frac{g(0) + (g'(0) - \lambda_1)b}{g(0) + (g'(0) - \lambda_1)m},$$

where $M = \sup_{x \in \Omega} \varphi(x) < b$, $m = \int_{\Omega} \varphi(x) \Psi_1(x) dx$.

3. TWO EXAMPLES

In this and the following sections, we investigate what would happen to the classical solution of problem (1.1) for Ω being an unbounded domain. To understand the whole spectrum of this problem, first let us investigate the following two examples.

EXAMPLE 1. For $\Omega = R^N$, we treat the initial value problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = g(s), & \text{in } R^N \times (0, T), \\ u(x, 0) = \varphi(x) & \text{in } R^N, \end{cases} \quad (3.1)$$

where $\varphi(x) \geq 0$, $\sup_{x \in R^N} \varphi(x) < b$, $g(s): [0, b) \mapsto (0, +\infty)$ satisfies (G1)–(G4).

Since the fundamental solution of the heat operator is a nonnegative operator, we know that the classical solution of problem (3.1) must satisfy

$$0 \leq u(x, t) < b \quad \text{in } R^N \times (0, T). \quad (3.2)$$

Let $u_0(x, t)$ be the classical solution of problem (3.1) with respect to $u(x, 0) \equiv 0$ and let $u_{\varphi}(x, t)$ be the classical solution of problem (3.1) with respect to $u(x, 0) = \varphi(x)$. Obviously one has

$$u_{\varphi}(x, t) \geq u_0(x, t) \quad \text{in } R^N \times (0, T^*), \quad (3.3)$$

where T^* is a real number such that $u_\varphi(x, t)$ and $u_0(x, t)$ exist for $t \in (0, T^*)$.

Now we choose a bounded domain $\Omega \subset R^N$ such that

$$\lambda_1(\Omega) < c_2 + \frac{c_1}{b},$$

and denote by $v(x, t)$ the classical solution of the following initial-boundary value problem:

$$\begin{cases} \frac{\partial v}{\partial t} - \Delta v = g(v) & \text{in } \Omega \times (0, T) \\ v = 0 & \text{on } \partial\Omega \times (0, T) \\ v(x, 0) = 0 & \text{in } \Omega. \end{cases}$$

On one hand, Theorem 2.1 implies that $v(x, t)$ quenches in finite time, and on the other hand, the maximum principle ensures that

$$u_0(x, t) \geq v(x, t), \quad \text{in } \Omega \times (0, T).$$

Consequently, $u_0(x, t)$ quenches in finite time. Hence, by (3.3) we know that $u_\varphi(x, t)$ quenches in finite time. This leads to the following.

Conclusion 3.1. The classical solution of problem (3.1) always quenches in finite time.

EXAMPLE 2. Let Ω be an infinite cone with vertex angle $\Theta \neq 0$. Investigate the initial-boundary value problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = g(u), & \text{in } \Omega \times (0, T), \\ u = 0, & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = \varphi(x), & \text{in } \Omega, \end{cases} \quad (3.4)$$

where $g(s)$ and $\varphi(x)$ are the same as in Example 1.

Since the vertex angle of Ω is not equal to zero, we can always choose a bounded domain $\Omega_0 \subset \Omega$ such that

$$\lambda_1(\Omega_0) < c_2 + \frac{c_1}{b}. \quad (3.5)$$

With (3.5) established, a discussion analogous to the proof of Conclusion 3.1 leads to the following.

Conclusion 3.2. The nonnegative classical solution of problem (3.4) always quenches in finite time.

Theorem 2.1, Conclusion 3.1, and Conclusion 3.2 seem to say that if Ω is an unbounded domain, then the existence of a nonnegative classical global solution of problem (1.1) is impossible. However, what surprises us is just that the nonnegative classical solution of problem (1.1) can still exist globally for some kinds of unbounded domains. To make this clear, in Section 4, we consider two kinds of domains

$$(I) \quad \Omega_I = R^{N-1} \times (-a, a)$$

$$(II) \quad \Omega_{II} = \Omega_0 \times (-\infty, \infty),$$

where a is a positive constant and $\Omega_0 \subset R^{N-1}$ is a bounded domain.

4. THE EXISTENCE OF A GLOBAL SOLUTION OF PROBLEM (1.1) FOR Ω AN INFINITE STRIP AND CYLINDER

In this section, we investigate problem (1.1) with $\Omega = \Omega_I$ or Ω_{II} . Our aim is to prove the following two theorems.

THEOREM 4.1. *Assume that $g(s)$ satisfies (G1) and (G2) and that $\Omega = \Omega_I$ in problem (1.1). If $u_\varphi^I(x, t)$ denotes the nonnegative classical solution of the above problem, then there are constants $a^* > a_M > 0$ such that*

- (i) $u_\varphi^I(x, t)$ exists globally for $a \in (0, a_M)$,
- (ii) $u_\varphi^I(x, t)$ quenches in finite time for $a > a^*$.

THEOREM 4.2. *Suppose that $g(s)$ satisfies (G1) and (G2) and that $\Omega = \Omega_{II}$ in problem (1.1). If $u_\varphi^{II}(x, t)$ denotes the nonnegative classical solution of the above problem, then we have that*

- (i) $u_\varphi^{II}(x, t)$ exists globally for Ω_0 small enough, and
- (ii) $u_\varphi^{II}(x, t)$ quenches in finite time for Ω_0 large enough.

Since the proof of Theorem 4.2 is similar to that of Theorem 4.1, we focus our attention on the proof of Theorem 4.1 and divide this proof into the following lemmas.

LEMMA 4.1. *There is at most one nonnegative classical solution of problem (1.1) with $\Omega = \Omega_I$ or Ω_{II} .*

Proof. Let $u_1(x, t)$, $u_2(x, t)$ be two arbitrary nonnegative classical solutions of problem (1.1). Set $w(x, t) = u_1(x, t) - u_2(x, t)$. A simple calculation implies that $w(x, t)$ satisfies

$$\begin{cases} \frac{\partial w}{\partial t} - \Delta w = d(x, t)w, & \text{in } \Omega \times (0, T), \\ w = 0, & \text{on } \partial\Omega \times (0, T), \\ w(x, 0) = 0, & \text{in } \Omega, \end{cases}$$

where $\Omega = \Omega_I$ or Ω_{II} .

By (G2), one has

$$0 \leq u_i(x, t) < b \quad \text{for } (x, t) \in \Omega \times (0, T), \quad i = 1, 2.$$

Thus $|w(x, t)| < b$ for $(x, t) \in \Omega \times (0, T)$. Now the conclusion of Lemma 4.1 follows immediately from the Pragmaen-Lindelöf principle (see [5]) since, by (G1), $d(x, t)$ is bounded for $t < T$.

To complete the Proof of Theorem 4.1, first let us investigate the problem

$$\begin{cases} \frac{\partial v}{\partial t} - \Delta v = g(v), & \text{in } \Omega_I \times (0, T), \\ v = 0, & \text{on } \partial\Omega_I \times (0, T), \\ v = 0, & \text{in } \Omega_I, \end{cases} \quad (4.1)$$

where $\Omega_I = R^{N-1} \times (-a, a)$, $\partial\Omega_I = \{(\tilde{x}, y) \mid \tilde{x} \in R^{N-1}, \text{ and } y \in \{-a, a\}\}$.

LEMMA 4.2. *Suppose that $g(s)$ satisfies (G1) and (G2) and $v(x, t)$ is the nonnegative classical solution of problem (4.1). Then there is a constant $a^* > 0$ such that $v(x, t)$ exists globally for $a \in (0, a^*)$ and quenches in finite time for $a > a^*$.*

Proof. For convenience, we represent $x \in \Omega_I$ as $x = (x_1, x_2, \dots, x_{N-1}, x_N) = (\tilde{x}, x_N) = (\tilde{x}, y)$. Let $v(x, t) = v(\tilde{x}, y, t)$ be a nonnegative classical solution of problem (4.1). Since the operator $(\partial/\partial t) - \Delta$ and the domain Ω_I are invariant under the translation of \tilde{x} , we know that, for any $h = (h_1, h_2, \dots, h_{N-1}) \in R^{N-1}$, $\tilde{v}(x, t) = v(\tilde{x} + h, y, t)$ is also a nonnegative classical solution of problem (4.1). Thus by Lemma 4.1 we must have $\tilde{v}(x, t) \equiv v(x, t)$, and hence $v(x, t) = v(\tilde{x}, y, t)$ depends only on y and t . This implies that problem (4.1) can be reduced to the following one-dimensional problem:

$$\begin{cases} \frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial y^2} = g(v), & \text{in } (-a, a) \times (0, T), \\ v(-a, t) = v(a, t) = 0, & \text{on } (0, T), \\ v(x, 0) = 0, & \text{in } (-a, a). \end{cases}$$

By a result of [4], we know that there is a constant $a^* > 0$ such that $v(y, t)$ exists globally for $a \in (0, a^*)$ and quenches in finite time for $a > a^*$. This completes the Proof of Lemma 4.2.

Second, we consider the problem

$$\begin{cases} \frac{\partial w}{\partial t} - \Delta w = g(w), & \text{in } \Omega_1 \times (0, T), \\ w = M, & \text{on } \partial\Omega_1 \times (0, T), \\ w = M, & \text{in } \Omega_1 \times \{0\}, \end{cases} \quad (4.2)$$

where Ω_1 and $\partial\Omega_1$ are the same as in problem (4.1), and $M = \sup_{x \in \Omega_1} \varphi(x) < b$.

LEMMA 4.3. *Assume that $g(s)$ satisfies (G1) and (G2) and $w(x, t)$ is the nonnegative classical solution of problem (4.2) then there is a constant $a_M > 0$ such that $w(x, t)$ exists globally for $a \in (0, a_M)$ and quenches in finite time for $a > a_M$.*

Proof. Let $R(x, t) = w(x, t) - M$. We can very easily verify that $R(x, t)$ satisfies

$$\begin{cases} \frac{\partial R}{\partial t} - \Delta R = \tilde{g}(R), & \text{in } \Omega_1 \times (0, T), \\ R = 0, & \text{on } \partial\Omega_1 \times (0, T), \\ R = 0, & \text{in } \Omega_1 \times \{0\}, \end{cases}$$

where $\tilde{g}(R) = g(M + R)$.

By (G1) and (G2), we have that $\tilde{g}(s)$ satisfies

($\tilde{G}1$) $\tilde{g}(s): [0, b - M) \mapsto (0, +\infty)$ is locally Lipschitz on $[0, b - M)$ and $\tilde{g}(0) = g(M) > 0$;

($\tilde{G}2$) $\lim_{s \rightarrow b - M} \tilde{g}(s) = \lim_{s \rightarrow b -} g(s) = +\infty$.

Thus, a discussion analogous to the Proof of Lemma 4.2 implies that there is a constant $a_M > 0$ such that $R(x, t)$ exists globally for $a \in (0, a_M)$ and quenches in finite time for $a > a_M$. Consequently, $w(x, t) = M + R(x, t)$ exists globally for $a \in (0, a_M)$ and quenches in finite time for $a > a_M$. This completes the Proof of Lemma 4.3.

Proof of Theorem 4.1. Let $u_\varphi^1(x, t)$ be the nonnegative classical solution of problem (1.1) with $\Omega = \Omega_1$, $v(x, t)$ be the nonnegative classical solution of problem (4.1), and $w(x, t)$ be the nonnegative classical solution of

problem (4.2). It is obvious that $v(x, t)$ is a subfunction of $u_\varphi^1(x, t)$ and $w(x, t)$ is a superfunction of $u_\varphi^1(x, t)$. Thus, by Lemmas 4.2 and 4.3, we have that

- (i) $u_\varphi^1(x, t)$ exists globally for $a \in (0, a_M)$, and
- (ii) $u_\varphi^1(x, t)$ quenches in finite time for $a > a^*$.

This completes the Proof of Theorem 4.1.

Remark. The Proof of Theorem 4.2 is similar to that of Theorem 4.1; hence we omit it here.

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